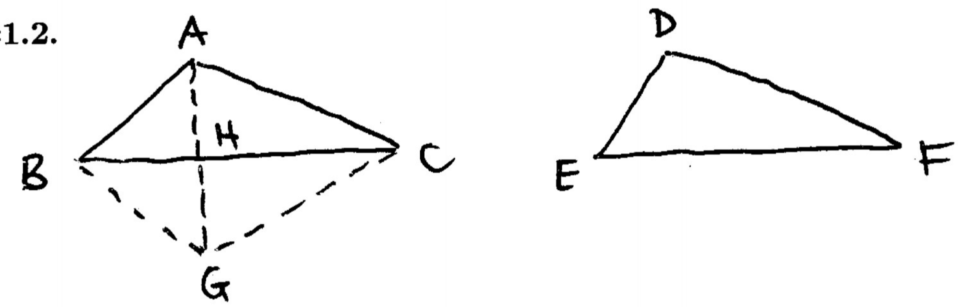
Solutions for Selected Problems in Geometry: Theorems and Constructions

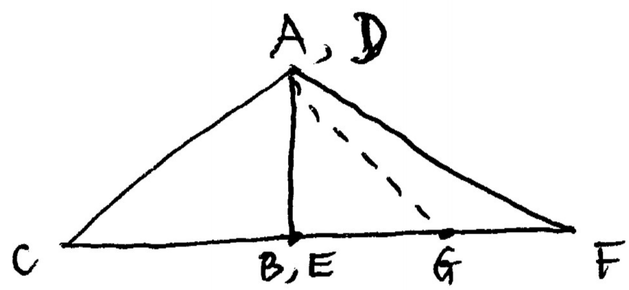
By
A. Berele and J. Goldman

p. 17 #1.2.



Given $\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$ in $\triangle ABC$ and $\triangle DEF$ respectively. We want to show that the two triangles are $\cong$. Construct G on the opposite side of $\overleftrightarrow{BC}$ from A so that $\angle GBC \cong \angle E$ and $\angle GCB \cong \angle F$. Work with the case where $\overline{AG}$ intersects $\overleftrightarrow{BC}$ in the point H, which is between B and C, as in the picture above. (The cases where $H = B, H = C$, H to the left of B, and H to the right of C are similar and must be done separately. We just do the one.) By ASA, we have $\triangle BGC \cong \triangle EDF$. From corresponding parts, $\overline{GC} \cong \overline{DF}$, and by transitivity, $\overline{AC} \cong \overline{GC}$. So, $\angle CAH \cong \angle CGH$. In like manner, obtain $\angle BAH \cong \angle BGH$. Add the angle measures to conclude $\angle BAC \cong \angle BGC$. Consequently, $\triangle BAC \cong \triangle BGC$, from SAS. Since $\cong$ is a transitive relation on $\triangle$, $\triangle BAC \cong \triangle EDF$ and we are finished.

p. 17, #1.3



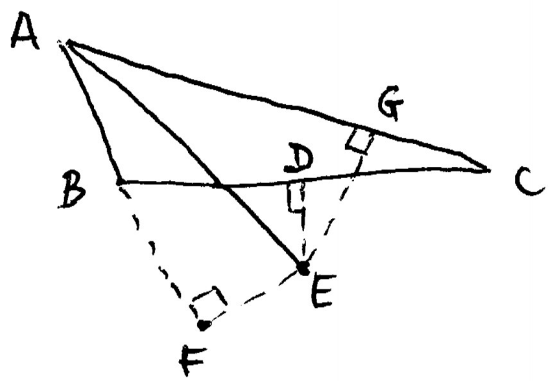
Assume the data of the problem. Because $\overline{AB} \cong \overline{DE}$, the triangles may be oriented as in the picture, A and D coincide, B and E coincide. $\angle ABC + \angle DEF = 180°$, so $\overleftrightarrow{CBF}$ is a line. From $\triangle ACF$ isosceles, obtain $\angle ACF \cong \angle AFC$. Claim: $\overline{CB} \cong \overline{BF}$; for, if not, one of CB or BF is smaller. Suppose $CB < BF$. Then there is a point G in the interior of $\overline{BF}$ such that $\overline{CB} \cong \overline{BG}$. But from SAS, $\triangle ABC \cong \triangle ABG$. $\therefore \angle AGC \cong \angle ACF \cong \angle AFC$. However, the exterior angle theorem implies $\angle AGC > \angle AFC$. Contradiction. Consequently $\overline{CB} \cong \overline{BF}$. $\therefore \triangle ABC \cong \triangle DEF$ by ASA.

p. 17, #1.4

Assume the data of the problem. Suppose $\overline{AB} \ncong \overline{DE}$. One of the last two segments is longer than the other, say $AB > DE$. Construct G on $\overline{BA}$ so that $\overline{BG} \cong \overline{ED}$. From SAS, $\triangle GBC \cong \triangle DEF$. $\therefore \angle BGC \cong \angle D \cong \angle A$. However, from the exterior angle theorem, $\angle BGC > \angle A$. Contradiction: Thus we must have $\overline{AB} \cong \overline{DE}$ and therefore $\triangle ABC \cong \triangle DEF$ by ASA.

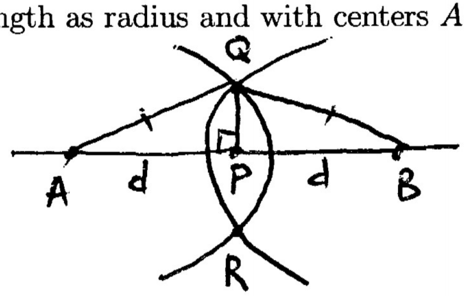
p. 17, #1.5

Consider the alternative diagram below. Note that E is not in the interior of $\triangle ABC$, that F is on $\overline{AB}$ extended, and that G is interior to $\overline{AC}$. As in the text, obtain $\overline{AF} \cong \overline{AG}$ and $\overline{BF} \cong \overline{GC}$; however, we must use subtraction to get $\overline{AB}$ and addition to get $\overline{AC}$. Thus, the "proof" will not work here. In fact, this is the situation in any scalene $\triangle$.



p. 17, #1.6

Set the points of your compass open to any convenient distance d and use it to determine points A and B on ℓ (by taking P as the center of a circle) such that $AP = PB = d$. Now, with compass points set on any convenient length apart $> d$, draw intersecting arcs of two circles, each with this length as radius and with centers A and B.



If Q is one of the intersection points of the two arcs, then one can show $\overleftrightarrow{PQ} \perp \ell$. The proof follows from the fact that $\triangle APQ \cong \triangle BPQ$ since we may use SSS. Further note that $\angle APQ = \angle BPQ$ and $\angle APQ + \angle BPQ = 180°$ imply that $\overleftrightarrow{PQ}$ meets ℓ at a $90°$ angle. Further congruence considerations yield the fact that the other intersection point, R, of the arcs lies on $\overleftrightarrow{PQ}$.