1 FUNCTIONS AND LIMITS

1.1 FUNCTIONS AND THEIR REPRESENTATIONS

SUGGESTED TIME AND EMPHASIS

2 classes Essential material

POINTS TO STRESS

1. Definition of function, including piecewise functions.
2. Understanding the interplay between the four ways of representing a function (verbally, numerically, visually, algebraically) perhaps using the concepts of increasing and decreasing functions as an example.
3. Finding the domain and range of a function, regardless of representation.
4. Investigating even and odd functions.

QUIZ QUESTIONS

- **TEXT QUESTION** Fill in the blanks: $|x| = \begin{cases} \_\_\_ & \text{if } x \geq 0 \\ \_\_\_ & \text{if } x < 0 \end{cases}$

  ANSWER x, $-x$

- **DRILL QUESTION** What is the domain of the function $f(x) = \sqrt{1 - \sqrt{x}}$?

  ANSWER $0 \leq x \leq 1$

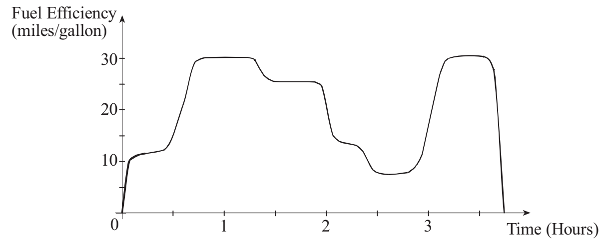
MATERIALS FOR LECTURE

- Draw a graph of electrical power consumption in the classroom versus time on a typical weekday, pointing out important features throughout, and using the vocabulary of this section as much as possible.
- In 1984, United States President Ronald Reagan proposed a plan to change the United States personal income tax system. According to his plan, the income tax would be 15% on the first \$19,300 earned, 25% on the next \$18,800, and 35% on all income above and beyond that. Describe this situation to the class, and have them graph (marginal) tax rate and tax owed versus income for incomes ranging from \$0 to \$80,000. Then have them try to come up with equations describing this situation.
- In the year 2000, Presidential candidate Steve Forbes proposed a "flat tax" model: 0% on the first \$36,000 and 17% on the rest. Have the students do the same analysis, and compare the two models. As an extension, perhaps have the students look at a current tax table and draw similar graphs.
- Let $f(x)$ be the leftmost nonzero digit of x. So $f(386.6) = 3$ and $f(0.000451) = 4$. Have the students try to find the domain and range of f.

  ANSWER The domain seems to be all real numbers except zero, and the range seems to be the set of integers from 1 through 9. Although the graph of this "function" cannot be drawn, ask the students to verify that it passes the Vertical Line Test. It turns out that it does not (and in fact is not even a function), for a subtle reason. For example, let $x = \frac{1}{5}$. If we write x as 0.2, then $f(x) = 2$, but if we write x as $0.1\overline{9}$, then $f(x) = 1$. Therefore, f is not a function.

WORKSHOP/DISCUSSION

- Draw a graph of fuel efficiency versus time on a trip, such as the one below. Lead a discussion of what could have happened on the trip.



- Present graphs of even and odd functions, such as $\sin x$, $\cos x + x^2$, and $\cos(\sin x)$, and check with the standard algebraic tests.
- Start with a table of values for the function $f(x) = \frac{1}{4}x^2 + x$:

x	0	1	2	3	4
$f(x)$	0	1.25	3	5.25	8

  First, have the class describe the behavior of the function in words, trying to elicit the information that the function is increasing, and that its rate of increase is also increasing. Then, have them try to extrapolate the function in both directions, debating whether or not the function is always positive and increasing. Plot the points and connect the dots, then have them try to concoct a formula (not necessarily expecting them to succeed).

- Discuss the domain and range of a function such as $f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

  Also talk about why f is neither increasing nor decreasing for $x > 0$. Stress that when dealing with new sorts of functions, it becomes important to know the precise mathematical definitions of such terms.

- Define "difference quotient" as done in Exercises 21–24. Define $f(x) = x^3$, and show that $\dfrac{f(a+h) - f(a)}{h} = 3a^2 + 3ah + h^2$. This example both reviews algebra skills and foreshadows future calculations.

GROUP WORK 1: EVERY PICTURE TELLS A STORY

Put the students in groups of four, and have them work on the exercise. If there are questions, encourage them to ask each other before asking you. After going through the correct matching with them, have each group tell their story to the class and see if it fits the remaining graph.

ANSWERS

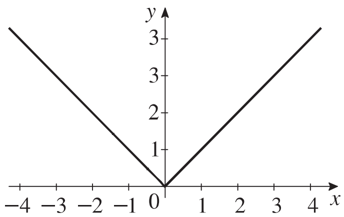
1. (b) **2.** (a) **3.** (c) **4.** The roast beef was cooked in the morning and put in the refrigerator in the afternoon.

GROUP WORK 2: A CHAIN OF FUNCTIONS

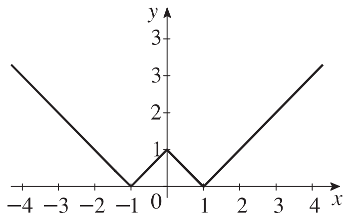
It is recommended that students not be allowed to use graphing technology to do this activity. The intention is to give them an opportunity to practice working with absolute values and order of operations, and to reinforce the idea of looking for mathematical patterns.

ANSWERS

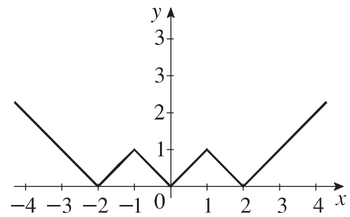
1.



$|x|$

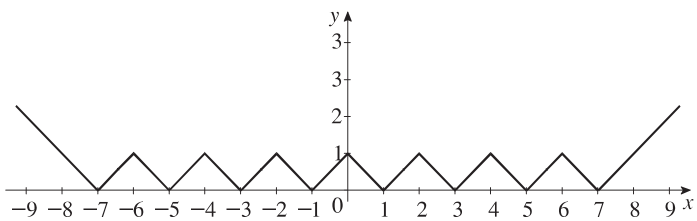


$|1 - |x||$



$|1 - |1 - |x|||$

2. (a) 1 (b)



$|1 - |1 - |1 - |1 - |1 - |1 - |1 - |x|||||||$

GROUP WORK 3: FINDING A FORMULA

Make sure that the students know the equation of a circle with radius r, and that they remember the notation for piecewise-defined functions. Split the students into groups of four. In each group, have half of the students work on each problem first, and then have them check each other's work. If the students find these problems difficult, have them work together on each problem.

ANSWERS

1. $f(x) = \begin{cases} -x - 2 & \text{if } x \leq -2 \\ x + 2 & \text{if } -2 < x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$ **2.** $g(x) = \begin{cases} x + 4 & \text{if } x \leq -2 \\ 2 & \text{if } -2 < x \leq 0 \\ \sqrt{4 - x^2} & \text{if } 0 < x \leq 2 \\ x - 2 & \text{if } x > 2 \end{cases}$

HOMEWORK PROBLEMS

CORE EXERCISES 6, 10, 18, 27, 34, 36, 50, 55, 65

SAMPLE ASSIGNMENT 6, 10, 14, 18, 22, 27, 32, 34, 36, 44, 47, 50, 52, 55, 58, 60, 62, 65

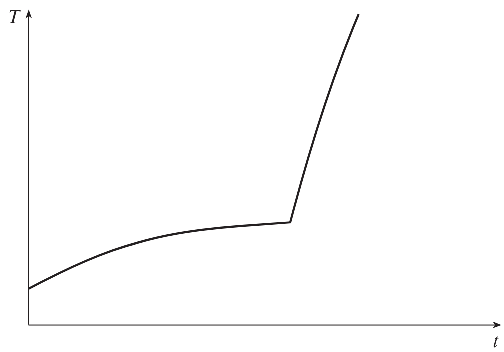
EXERCISE	D	A	N	G
6				×
10	×			×
14				×
18				×
22		×		
27		×		

EXERCISE	D	A	N	G
32		×		×
34		×		×
36		×		×
44		×		
47		×		
50		×		

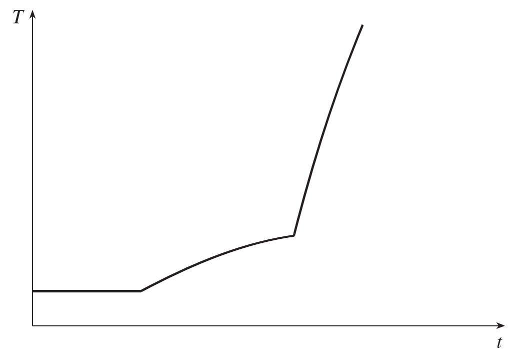
EXERCISE	D	A	N	G
52		×		×
55				×
58				×
60		×		
62		×		
65	×			

GROUP WORK 1, SECTION 1.1
Every Picture Tells a Story

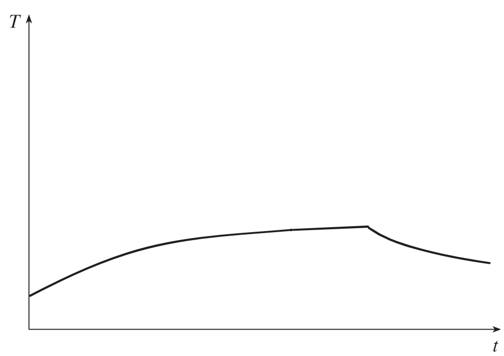
One of the skills you will be learning in this course is the ability to take a description of a real-world occurrence, and translate it into mathematics. Conversely, given a mathematical description of a phenomenon, you will learn how to describe what is happening in plain language. Here follow four graphs of temperature versus time and three stories. Match the stories with the graphs. When finished, write a similar story that would correspond to the final graph.



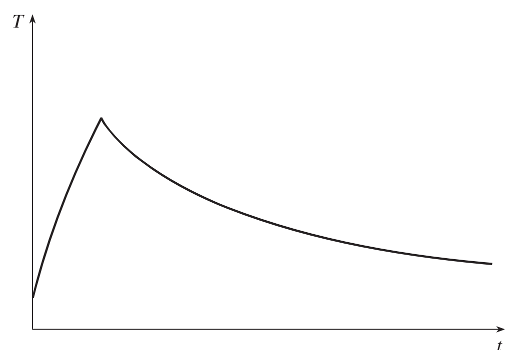
Graph 1



Graph 2



Graph 3



Graph 4

(a) I took my roast beef out of the freezer at noon, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

(b) I took my roast beef out of the freezer this morning, and left it on the counter to thaw. Then I cooked it in the oven when I got home.

(c) I took my roast beef out of the freezer this morning, and left it on the counter to thaw. I forgot about it, and went out for Chinese food on my way home from work. I put it in the refrigerator when I finally got home.